On the spectral behavior of the Neumann Laplacian under mass density perturbation

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Introduction

Let Ω be a domain in $\mathbb{R}^N$ of finite measure.

Let $\rho \in \mathcal{R} := \{f \in L^\infty(\Omega) : \operatorname{ess\,inf}_\Omega f(x) > 0\}$

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$$\mathcal{L}u = \sum_{0 \leq |\alpha|, |\beta| \leq m} (-1)^\alpha D^\alpha \left(A_{\alpha\beta} D^\beta u\right)$$

and the eigenvalue problem

$$\mathcal{L}u = \lambda \rho u$$

subject to homogeneous boundary conditions (Dirichlet, Neumann, intermediate, etc.)

Introduction

$$\mathcal{Q}[u,\varphi] := \int_\Omega \sum_{0\leq|\alpha|,|\beta|\leq m} A_{\alpha\beta} D^\alpha u D^\beta \varphi dx = \lambda \int_\Omega u\varphi\rho dx \ \ \forall \varphi \in V(\Omega)$$

$$\mathcal{Q}[u,\varphi] := \int_\Omega \sum_{0\le|\alpha|,|\beta|\le m} A_{\alpha\beta} D^\alpha u D^\beta \varphi dx = \lambda \int_\Omega u\varphi\rho dx \ \ \forall \varphi \in V(\Omega)$$

- $V(\Omega) \subset H^m(\Omega)$ closed with $V(\Omega) \subset L^2(\Omega)$ compact;
- $A_{\alpha\beta} \in L^\infty(\Omega)$ with $A_{\alpha\beta} = A_{\beta\alpha}$;
- there exist $a, b, c > 0$ such that

$$a\|u\|^2_{H^m(\Omega)} \le \mathcal{Q}[u,u] + b\|u\|^2_{L^2(\Omega)},$$
$$\mathcal{Q}[u,u] \le c\|u\|^2_{H^m(\Omega)};$$

Poly-harmonic operators

$$(-\Delta)^m u = \lambda \rho u$$

Let $0 \leq k \leq m$ and $V(\Omega) = H^m(\Omega) \cap H_0^k(\Omega)$.

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- $k = m$ Dirichlet boundary conditions, $V(\Omega) = H_0^m(\Omega)$ (N=2, m=2 clamped plate);
- $0 < k < m$ Intermediate boundary conditions, $V(\Omega) = H^m(\Omega) \cap H_0^k(\Omega)$ (N=2, m=2 hinged plate);
- $k = 0$ Neumann-type boundary conditions, $V(\Omega) = H^m(\Omega)$ (N=2, m=2 free vibrating plate).

The eigenvalue problem

Our problem has a divergent sequence of eigenvalues

$$-b < \lambda_1[\rho] \leq \lambda_2[\rho] \leq \cdots \leq \lambda_j[\rho] \leq \cdots$$

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Our aim is to study the dependence

$$\rho \mapsto \lambda_j[\rho]$$

Analiticity of the eigenvalues

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Theorem

Let F be a nonempty finite subset of $\mathbb{N}$ and let

$$\mathcal{R}[F] := \{\rho \in \mathcal{R} \, : \, \lambda_j[\rho] \neq \lambda_l[\rho] \, , \ \forall j \in F, l \in \mathbb{N} \setminus F\} \ ,$$
$$\Theta[F] := \{\rho \in \mathcal{R}[F] \, : \, \lambda_{j_1}[\rho] = \lambda j_2[\rho] \, , \ \forall j_1, j_2 \in F\}.$$

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Then $\mathcal{R}[F]$ is open in $L^\infty(\Omega)$ and the symmetric functions of the eigenvalues

$$\Lambda_{F,h}[\rho] = \sum_{\substack{j_1,\dots,j_h \in F \\ j_1 < \dots < j_h}} \lambda_{j_1}[\rho] \cdots \lambda_{j_h}[\rho]\,, \quad h = 1, \dots, |F|$$

are analytic in $\mathcal{R}[F]$.

Derivatives of the eigenvalues

Theorem

Let F be a nonempty finite subset of $\mathbb{N}$. If $F = \cup_{k=1}^{n} F_k$ and $\rho \in \cap_{k=1}^{n} \Theta[F_k]$ is such that for each $k = 1, ..., n$ the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_{F_k}[\rho]$ for all $j \in F_k$, then the differential of $\Lambda_{F,h}$ at ρ is given by the formula

$$d\Lambda_{F,h}[\rho][\dot{\rho}] = -\sum_{k=1}^{n} c_k \sum_{l \in F_k} \int_{\Omega} (u_l)^2 \dot{\rho}\, dx \,,$$

for all $\dot{\rho} \in L^{\infty}(\Omega)$, where for each $k = 1, ..., n$, $\{u_l\}_{l \in F_k}$ is an orthonormal basis in $L^2_{\rho}(\Omega)$ of the eigenspace associated with $\lambda_{F_k}[\rho]$.

Critical mass densities

We assume $V(\Omega) \subset H^1_0(\Omega)$

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Theorem

Let F be a nonempty finite subset of $\mathbb{N}$. Then for all $h = 1, ..., |F|$ the function which takes $\rho \in \mathcal{R}[F] \cap L_M$ to $\Lambda_{F,h}[\rho]$ has no critical mass densities $\tilde{\rho}$ such that $\lambda_j[\tilde{\rho}] \neq 0$ and have the same sign for all $j \in F$.

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$$\sum_{k=1}^{n} c_k \sum_{l \in F_k} u_l^2 = \mathrm{const} \Longrightarrow u_1 = ... = u_{|F|} = 0$$

Maximum principle

Theorem

Let $C \subset L^\infty(\Omega)$ be a bounded set. Then the functions from C to $\mathbb{R}$ which take $\rho \in C$ to $\lambda_j[\rho]$ are weakly continuous for all $j \in \mathbb{N}$.*

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Theorem

Let $C \subseteq \mathcal{R}[F]$ be a weakly compact subset of $L^\infty(\Omega)$. Let $M > 0$ such that $C \cap L_M$ is not empty. Assume that all the eigenvalues $\lambda_j[\rho]$ have the same sign and do not vanish for all $j \in \mathbb{N}$, $\rho \in C$. Then for all $h = 1, ..., |F|$ the function which takes $\rho \in C \cap L_M$ to $\Lambda_{F,h}[\rho]$ has maxima and minima, and such points belong to $\partial C \cap L_M$.*

Neumann boundary conditions

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1.

The eigenvalue problem for the Laplacian with Neumann boundary conditions is

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

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We have a sequence

$$0 < \lambda_1[\rho] \leq \lambda_2[\rho] \leq \cdots \leq \lambda_j[\rho] \leq \cdots$$

Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, $F = \{m, n\}$, with $m, n \in \mathbb{N}$, $m \neq n$. Let $\tilde{\rho} \in \mathcal{R}[F]$ continuous, such that the solutions of (1) be classic solutions and moreover their nodal domains are stokians. Then for $h = 1, 2$, $\tilde{\rho}$ is not a critical mass density for the function which takes $\rho \in \mathcal{R}[F] \cap L_M$ to $\Lambda_{F,h}[\rho]$. Moreover all simple eigenvalues have no critical mass densities under the fixed mass constraint.

$$\sum_{i \in F} c_i u_i^2 = \mathrm{const}$$

Critical mass densities

Theorem

Let $\Omega \subset \mathbb{R}^N$ and F be as in Theorem 6. Let $C \subseteq \mathcal{R}[F]$ be a weakly compact subset of $L^\infty(\Omega)$. Let $M > 0$ and $L_M = \{\rho \in L^\infty(\Omega) : \int_\Omega \rho = M\}$. Then for $h = 1, 2$, the function which takes $\rho \in C \cap L_M$ to $\Lambda_{F,h}[\rho]$ admits points of maximum and minimum, and if for such points the solutions of problem (1) are classic solution, they belong to $\partial C \cap L_M$.*

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$$\begin{cases} \Delta u = 0, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = \lambda \rho u, & \text{on } \partial\Omega\,. \end{cases} \tag{2}$$

$\rho \in \mathcal{R}' := \{f \in L^\infty(\partial\Omega) : \operatorname{ess\,inf}_{\partial\Omega} f(x) > 0\}$.

Steklov boundary conditions

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We have a sequence

$$0 < \lambda_1[\rho] \leq \lambda_2[\rho] \leq \cdots \leq \lambda_j[\rho] \leq \cdots$$

Analyticity of eigenvalues and derivatives

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Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and F a nonempty finite subset of $\mathbb{N}$. Then the symmetric functions of eigenvalues $\Lambda_{F,h}$ are analytic in $\mathcal{R}[F]$.
Moreover, if $\rho \in \Theta[F]$ and the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_F[\rho]$ for all $j \in F$, then the differential of $\Lambda_{F,h}$ at ρ is given by the formula

$$d\Lambda_{F,h}[\rho][\dot{\rho}] = -\left(\lambda_F[\rho]\right)^{h+1} \binom{|F|-1}{h-1} \sum_{l \in F} \int_{\partial\Omega} u_l^2 \,\dot{\rho}\, d\sigma\,,$$

for all $\dot{\rho} \in L^\infty(\partial\Omega)$, where $\{u_l\}$ is a hortonormal basis for $\lambda_F[\rho]$ in $H^{1,0}_\rho(\Omega) := \left\{u \in H^1(\Omega) : \int_{\partial\Omega} u\rho d\sigma = 0\right\}$.

Critical mass densities

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Proposition

Let $B = B^N(0,1)$ be the unit ball in $\mathbb{R}^N$, S_N the $(N-1)$-dimensional measure of ∂B, $F = \{1, ..., N\}$, $M > 0$. Then the constant mass density $\rho_M = \frac{M}{S_N}$ is a critical mass density for $\Lambda_{F,h}$ for $h = 1, ..., N$ under the constraint $\int_{\partial\Omega} \rho\sigma = M$.

Critical mass densities

Theorem (C. Bandle 1968)

Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain of symmetry order q and suppose that the mass density ρ satisfies the symmetry condition $\rho(e^{\frac{2\pi i}{q}} z) = \rho(z)$ on $\partial\Omega$. Then

$$\lambda_{2n-1}[\rho], \lambda_{2n}[\rho] \leq \frac{2\pi n}{M}, \quad 1 \leq n \leq \frac{q-1}{2}, \quad \text{if } q \text{ odd},$$

$$\lambda_{2n-1}[\rho], \lambda_{2n}[\rho] \leq \frac{2\pi n}{M}, \quad 1 \leq n \leq \frac{q-2}{2},$$
$$\lambda_{q-1}[\rho] \leq \frac{\pi q}{m} \quad \text{if } q \text{ even}$$

and the equality is attained at the circle with constant mass density.

Remark

Let $B = B(0,1) \subset \mathbb{R}^2$ *and*

$$\rho(\theta) = \frac{M}{2\pi} + \sum_{j=1}^{+\infty} a_j \sin(j\theta) + b_j \cos(j\theta).$$

Then

$$\lambda_1[\rho] \leq \frac{2\pi}{M}$$

for all $\rho \in \mathcal{R}'$ *with* $\int_{\partial B} \rho = M$ *such that* $b_1 = b_2 = 0$. *The equality is attained at the constant density.*

Neumann vs Steklov

Let $B = B(0,1)$ be the unit ball in $\mathbb{R}^N$, $M > 0$, ω_N the volume of B, S_N the $(N-1)$-dimensional measure of ∂B. Let B_ε be the ball $B(0, 1-\varepsilon)$. Let $\rho_\varepsilon \in \mathcal{R}$ be defined by

$$\rho_\varepsilon(x) := \begin{cases} \varepsilon, & \text{if } x \in B_\varepsilon, \\ \tilde{\rho}_\varepsilon = \frac{M - \varepsilon\omega_N(1-\varepsilon)^N}{\omega_N\left(1-(1-\varepsilon)^N\right)}, & \text{if } x \in B \setminus B_\varepsilon, \end{cases} \tag{3}$$

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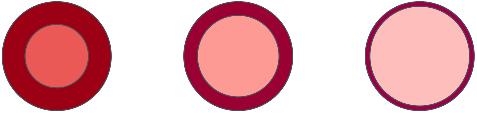

Spectral convergence

$$\frac{\int_B |\nabla u|^2 dx}{\int_B \rho_\varepsilon u^2 dx} \to \frac{\int_B |\nabla u|^2 dx}{\frac{M}{S_N}\int_{\partial B} u^2 dx}$$

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We proved compact convergence of resolvent operators, which implies norm convergence.

Theorem

Let $B = B(0,1)$ be the unit ball in $\mathbb{R}^N$, $M > 0$, S_N the $(N-1)$-dimensional measure of ∂B and $\rho_\varepsilon \in \mathcal{R}$ be defined as in (22). Let $\lambda_j[\rho_\varepsilon]$ be the eigenvalues of problem (1) on B for all $j \in \mathbb{N}$. Let $\overline{\lambda}_j$ be the eigenvalues of (2) on B corresponding to the constant density $\frac{M}{S_N}$. Then for all $j \in \mathbb{N}$ we have $\lim_{\varepsilon \to 0} \lambda_j[\rho_\varepsilon] = \overline{\lambda}_j$.

Spectral convergence

Theorem

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2, $M > 0$. We denote by Ω_ε the set $\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$. Let $\rho_\varepsilon \in \mathcal{R}$ be defined by

$$\rho_\varepsilon(x) := \begin{cases} \varepsilon, & \text{if } x \in \Omega_\varepsilon, \\ \frac{M - \varepsilon|\Omega_\varepsilon|}{|\Omega \setminus \Omega_\varepsilon|}, & \text{if } x \in \Omega \setminus \Omega_\varepsilon, \end{cases}$$

Let $\lambda_j[\rho_\varepsilon]$ be the eigenvalues of problem (1) for all $j \in \mathbb{N}$. Let $\overline{\lambda}_j$ be the eigenvalues of problem (2) corresponding to the constant mass density $\frac{M}{|\partial\Omega|}$. Then for all $j \in \mathbb{N}$ we have $\lim_{\varepsilon \to 0} \lambda_j[\rho_\varepsilon] = \overline{\lambda}_j$.

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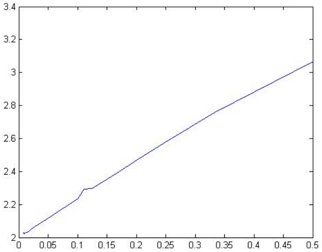
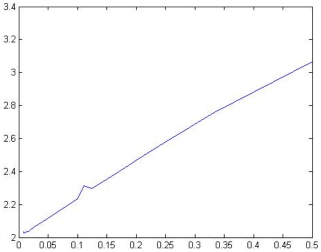
Numerical experiments on $B(0,1)$ in $\mathbb{R}^2$ and $M = \pi$. The first and second eigenvalues for the Steklov problem with constant surface density $\rho_\pi \equiv \frac{1}{2}$ on ∂B are $\lambda_1 = \lambda_2 = 2$.



(a) λ_1



(b) λ_2

Let $B = B(0,1) \subset \mathbb{R}^N$. Let $u(r,\theta) = \tilde{u}(r)\phi_l(\theta)$, where

$$\tilde{u}(r) := \begin{cases} r^{1-\frac{N}{2}} J_{\nu_l}(\sqrt{\lambda\varepsilon} r), & \text{if } r \leq 1-\varepsilon, \\ r^{1-\frac{N}{2}} (\alpha J_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon} r) + \beta Y_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon} r)), & \text{if } 1-\varepsilon < r < 1. \end{cases}$$

Here $\nu_l = \frac{(N+2l-2)}{2}$ for $l \in \mathbb{N}$, $\phi_l(\theta) = \phi_l(\theta_1, \ldots \theta_{N-1})$ is a solution of

$$-\delta\phi_l = l(l+N-2)\phi_l$$

and $-\delta$ is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$.

Derivatives at $\varepsilon = 0$

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We impose continuity of $\tilde{u}(r)$ and $\tilde{u}'(r)$ at $r = 1 - \varepsilon$ to get α, β.

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We impose Neumann boundary conditions $\tilde{u}'(r)_{|_{r=1}} = 0$ and we get

$$F(\lambda, \varepsilon) = 0$$

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We impose Neumann boundary conditions $\tilde{u}'(r)_{|_{r=1}} = 0$ and we get

$$F(\lambda, \varepsilon) = 0$$

Consider $\lambda[\varepsilon]$ and $\lambda'[\varepsilon]$. We used Talylor expansions of F and recursive formulas for the cross products of Bessel Functions and their derivatives. Finally we let $\varepsilon \to 0$

$$\lambda[0] = \frac{lN\omega_N}{M}, \tag{4}$$

$$\lambda'[0] = \frac{2l\lambda[0]}{3} + \frac{2\lambda^2[0]}{N(2l+N)}, \tag{5}$$

in particular $\lambda'[0] > 0$ for all $M > 0, N \geq 2, l \in \mathbb{N}$.

Final Remarks

In order to complete the picture:

- Non-existence of critical mass densities for the eigenvalues with Neumann boundary conditions;
- What kind of critical point is the constant density for the Steklov eigenvalues: in [1, Bandle] it is stated that it is indeed a maximum if restricted to a subset of the densities we considered;
- Extending the results of [1, Bandle] for $N > 2$.
- Formulas of derivatives at $\varepsilon = 0$ of the eigenvalues for more general domains Ω;
- Consider these problems for poly-harmonic operators ($(-\Delta)^m$ with Neumann and Steklov boundary conditions).

Essential bibliography

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Complements: Derivatives at $\varepsilon = 0$

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Set $a = (1-\varepsilon)\sqrt{\varepsilon\lambda}$, $b = (1-\varepsilon)\sqrt{\lambda\tilde{\rho}_\varepsilon}$. We impose continuity of $\tilde{u}(r)$ and $\tilde{u}'(r)$ at $r = 1-\varepsilon$ to get α, β.

$$\alpha = \frac{\pi}{2}\big(bJ_{\nu_l}(a)Y'_{\nu_l}(b) - aJ'_{\nu_l}(a)Y_{\nu_l}(b)\big),$$
$$\beta = \frac{\pi}{2}\big(aJ_{\nu_l}(b)J'_{\nu_l}(a) - bJ'_{\nu_l}(b)J_{\nu_l}(a)\big).$$

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We impose Neumann boundary conditions $\tilde{u}'(r)_{|_{r=1}} = 0$.

Complements: Derivatives at $\varepsilon = 0$

$$
\begin{aligned}
F(\lambda,\varepsilon) &= (1-\frac{N}{2})\Big[J_{\nu_l}(a)\big(Y'_{\nu_l}(b)J_{\nu_l}(\frac{b}{1-\varepsilon}) - J'_{\nu_l}(b)Y_{\nu_l}(\frac{b}{1-\varepsilon})\big)\\
&+ \frac{a}{b}J'_{\nu_l}(a)\big(J_{\nu_l}(b)Y_{\nu_l}(\frac{b}{1-\varepsilon}) - Y_{\nu_l}(b)J_{\nu_l}(\frac{b}{1-\varepsilon})\big)\Big]\\
&+ \frac{b}{(1-\varepsilon)}\Big[J_{\nu_l}(a)\big(Y'_{\nu_l}(b)J'_{\nu_l}(\frac{b}{1-\varepsilon}) - J'_{\nu_l}(b)Y'_{\nu_l}(\frac{b}{1-\varepsilon})\big)\\
&+ \frac{a}{b}J'_{\nu_l}(a)\big(J_{\nu_l}(b)Y'_{\nu_l}(\frac{b}{1-\varepsilon}) - Y_{\nu_l}(b)J'_{\nu_l}(\frac{b}{1-\varepsilon})\big)\Big] = 0.
\end{aligned}
$$

The hypothesis of Implicit Function Theorem are fulfilled and we have implicitly the eigenvalues as functions of ε.

Complements: Derivatives at $\varepsilon = 0$

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Consider $\lambda[\varepsilon]$ and $\lambda'[\varepsilon]$. We used Talylor expansions of F and recursive formulas for the cross products of Bessel Functions and their derivatives. Finally we let $\varepsilon \to 0$

$$\lambda[0] = \frac{lN\omega_N}{M}, \quad (6)$$

$$\lambda'[0] = \frac{2l\lambda[0]}{3} + \frac{2\lambda^2[0]}{N(2l+N)}, \quad (7)$$

in particular $\lambda'[0] > 0$ for all $M > 0, N \geq 2, l \in \mathbb{N}$.

Complements

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Definition

Let H be a real Hilbert space, $\mathcal{K}(H,H)$ the Banach subspace of $\mathcal{L}(H,H)$ of those $T \in \mathcal{L}(H,H)$ which are compact. A set $\mathcal{K} \subset \mathcal{K}(H,H)$ is said to be collectively compact if and only if the set $\{K[x] : K \in \mathcal{K}, x \in B\}$, where B is the unit ball in H, has compact closure. We say that a sequence of compact operators $\{K_n\}_{n\in\mathbb{N}}$ compactly converges to the compact operator K if $\{K_n\}_{n\in\mathbb{N}}$ is collectively compact and $K_n[x_n] \to K[x]$ whenever $x_n \to x$ in H.

Theorem

Let H be a real Hilbert space, $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{K}(H,H)$ compactly convergent to $K \in \mathcal{K}(H,H)$, and K_n and K are self-adjoint for all $n \in \mathbb{N}$. Then

$$\lim_{n\to+\infty} \|K_n - K\|_{\mathcal{L}(H,H)} = 0.$$

Definition

A domain $\Omega \subset \mathbb{R}^2$ is said to be of symmetry order q if there exists a symmetry center O such that Ω is invariant with respect to a rotation of an angle $\frac{2\pi}{q}$ around O.

Bilaplacian with Neumann conditions

1. n=2, N=2

$$\begin{cases} (-\Delta)^2 u = \lambda \rho u, & \text{in } \Omega\,, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega\,, \\ \frac{d}{ds}\frac{\partial^2 u}{\partial \nu \partial t} + \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega\,; \end{cases}$$

2. n=2, N>2

$$\begin{cases} (-\Delta)^2 u = \lambda \rho u, & \text{in } \Omega\,, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega\,, \\ \mathrm{div}_{\partial\Omega}\left(\mathrm{P}_{\partial\Omega}\left[(D^2 u).\nu\right]\right) + \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega\,; \end{cases}$$

Complements

A bounded open set Ω in $\mathbb{R}^N$ is called stokian if its regular boundary $\partial_{\mathrm{reg}}\Omega$ has finite $(N-1)$ dimensional measure and $\partial\Omega \setminus \partial_{\mathrm{reg}}\Omega$ has zero $(N-1)$ dimensional measure.